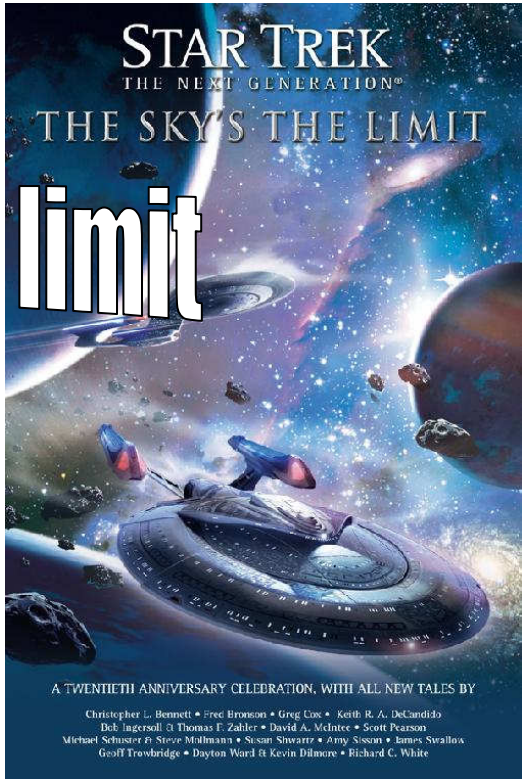
Nonlinearity challenges quantum metrology limits

Or

maybe the sky is the limit



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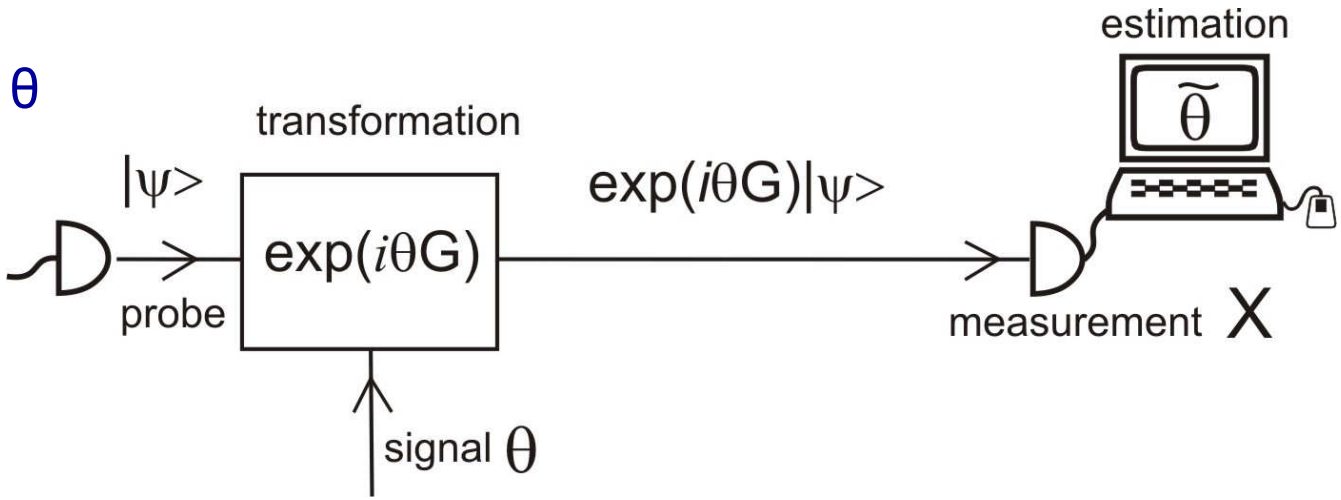
Introduction (4 transparencies)

detecting a signal θ



Universal structure of any signal detection:

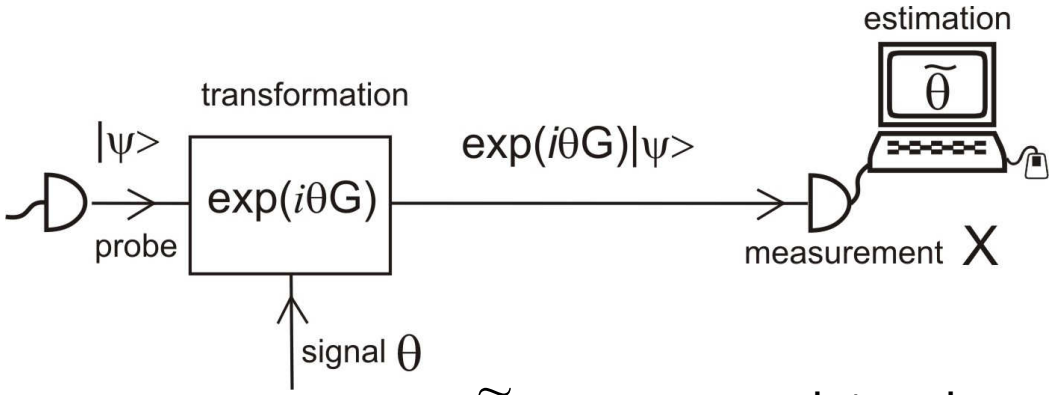
A probe in a known state ψ experiences a signal-dependent transformation.
The change of the probe state is monitored by a measurement X
The outputs allow us to estimate the unknown value of the signal.

Linear versus nonlinear transformations (single-mode picture)

$$G = a^{+}a \qquad\qquad G = \left(a^{+}a\right)^{k>1}$$

Main objective:

to obtain estimates as close as possible to the true value of the signal $\tilde{\theta}(x)$
with minimum uncertainty $\Delta\tilde{\theta}$



key point: uncertainty $\Delta\tilde{\theta}(X,G,\psi)$ determines resolution and allows us to compare different schemes

Several performance estimators: Fisher information Cramér-Rao bound

$$\Delta\tilde{\theta} = \frac{1}{\sqrt{NF}} \qquad F = \int dx \frac{1}{P(x\,|\,\theta)}\left(\frac{dP(x\,|\,\theta)}{dx}\right)^2$$

N = number of repetitions of the measurement
F = Fisher information
P = Statistics

To be completed with:
***efficiency* (how many N are necessary to reach the Cramér-Rao bound?)**
***prior information* (does detection actually improves knowledge of the signal?)**

Fixed mean number of photons $\text{total} = \bar{n}$

N = repetitions of the measurement

$\bar{n}$ = mean number of photons in the probe $\bar{n} = \langle \psi | a^{+} a | \psi \rangle$

Heisenberg limit = Best believed resolution for linear generators $G = a^{+} a$

two versions

***weak* Heisenberg limit**

$$\Delta\tilde{\theta}_{\min} = \frac{1}{2\sqrt{\bar{n}}}$$

***strong* Heisenberg limit**

$$\Delta\tilde{\theta}_{\min} = \frac{1}{2\ \text{total}}$$

$$\frac{1}{\sqrt{\bar{n}}} \geq \frac{1}{\bar{n}}$$

In practice reached when

$= 1,\ \bar{n} = \text{total}$

Both reached with probes in nonclassical states

strong form not practical:

currently nonclassical states have small photon numbers

say $\bar{n} \approx 10$

First part

Nonlinear schemes: better than both Heisenberg limits even with probes in robust and intense (plenty of photons) classical states

$$G = \left(a^{+}a\right)^{2} \qquad \Delta\widetilde{\theta}_{\min} << \frac{1}{2\quad\bar{n}}$$

Second part

Fisher information is nonlinear on resources $F \propto \bar{n}^{2}$

This allows to reach strong Heisenberg limit for large number N of runs in **linear schemes** (very appropriate for weak nonclassical probes)

$$G = a^{+}a \qquad \Delta\widetilde{\theta}_{\min} \approx \frac{1}{2\quad\bar{n}}, \text{ for } \quad >> 1$$

Third part

Alternative assessments or uncertainty (≠ variance): contradictions and lack of limits? even for linear schemes

First part (5 transparencies)

Nonlinear schemes: better than any Heisenberg limit even with probes
In robust classical states

$$G = \left(a^{+} a\right)^{2} \qquad \Delta\tilde{\theta}_{\min} << \frac{1}{2 \quad \bar{n}}$$

Usually nonlinear processes are used to prepare probes in optimum nonclassical probes required to reach the Heisenberg limits.

Better results are obtained using nonlinearity to imprint the signal on classical probes.

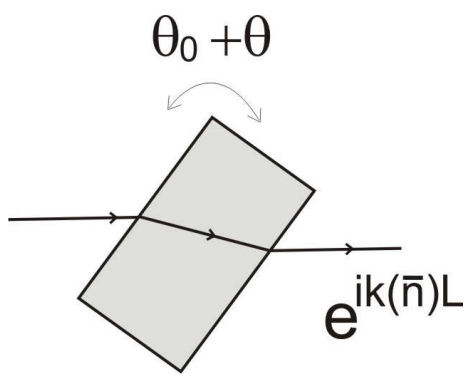
Nonlinear generators: propagation in optically nonlinear media

simple illustration: rotation sensor = Nonlinear gyroscope

Rotation is detected by the phase change experienced by a light wave.

The phase shift depends on the optical path *L*. weak signal $L \propto \theta$

Because of nonlinearity the phase change depends on the intensity of the input wave.



nonlinear optics **for example Kerr effect**

$$k(\bar{n}) \propto \bar{n}, \bar{n}^2, \ldots$$

$$\text{phase shift} \propto \bar{n}\,\theta$$

Nonlinearity can amplify the signal

Nonlinear generators (Kerr effect) + classical probe $G = \left(a^{+} a\right)^{2}$

probe in a coherent state: $\Delta\tilde{\theta} \propto \frac{1}{1/2\,\bar{n}^{3/2}}$

weak Heisenberg limit

$$\Delta\tilde{\theta}_{\min} = \frac{1}{2\sqrt{\bar{n}}}$$

strong Heisenberg limit

$$\Delta\tilde{\theta}_{\min} = \frac{1}{2\,\bar{n}}$$

Weak and strong Heisenberg limits can be beaten with nonlinear schemes and classical probes

Key points:

$\bar{n} >> 1$ **and robustness of classical states**

Experimental demonstration

Interaction-based quantum metrology showing scaling beyond the Heisenberg limit

M. Napolitano[1], M. Koschorreck[1], B. Dubost[1,2], N. Behbood[1], R. J. Sewell[1] & M. W. Mitchell[1]

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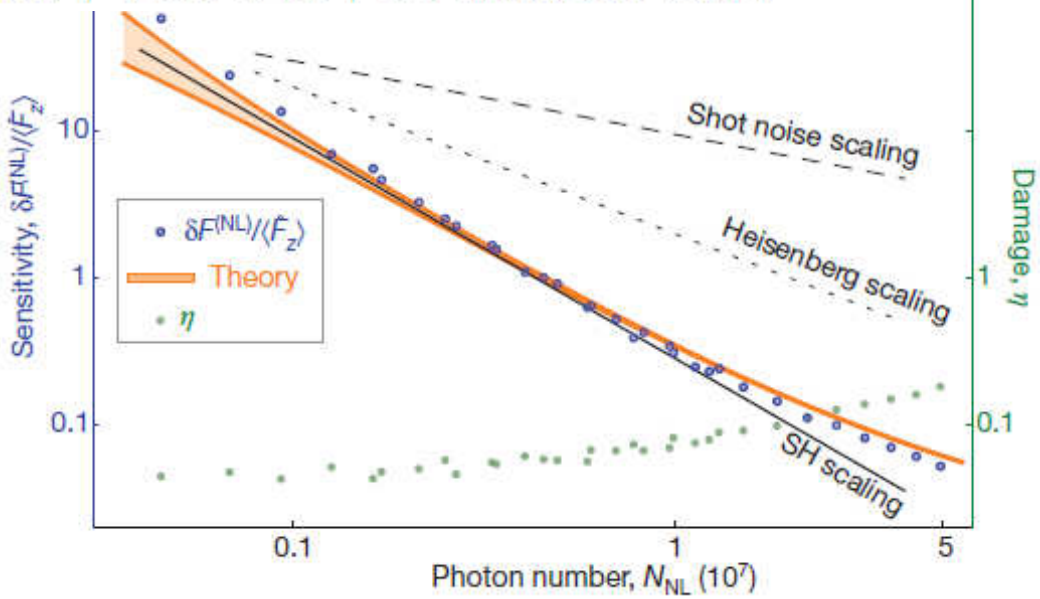
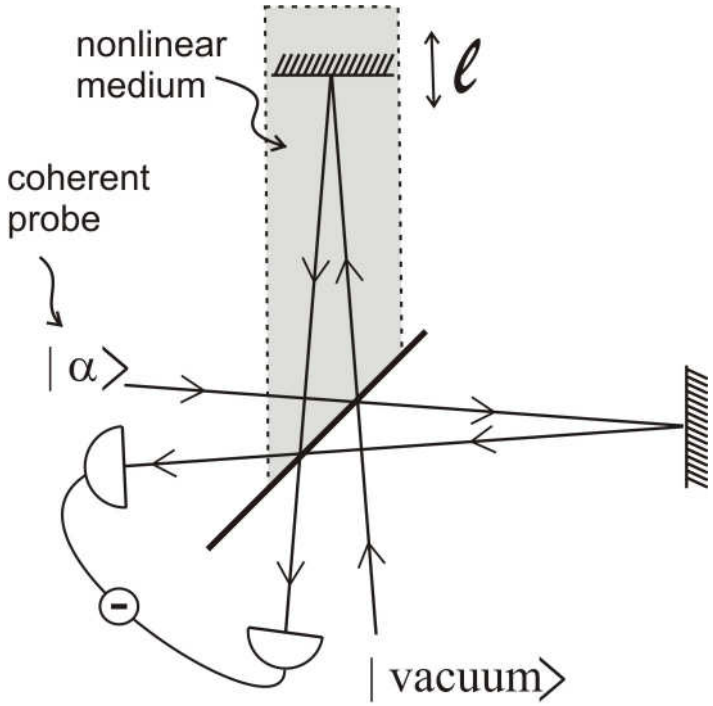


Other proposals: two-body interactions on Bose-Einstein condensates

Boixo, Davis, Tacla, Flammia, Caves, Geremia, Rey, Jiang, Lukin, Datta, Shaji, Bagan, Choi, Sundaram….

Further example:
detection of mirror displacements ℓ in a nonlinear Michelson interferometer illuminated with light pulses of n photons and duration τ



phase randomization

photons per pulse

thermal photons

quantum efficiency

$$\Delta\ell = \frac{\sqrt{1+\eta\sigma^2\bar{n}+n_t}}{1+\dfrac{n_2\hbar\omega\bar{n}}{n_0 A\tau}}\,\Delta\ell_{\text{linear}}$$

nonlinear index

linear index

pulse transverse width

pulse duration

Resolution depends not only on *n* but also on *A* and τ

new variables → new limits, new devices

Brief relaxation
while shifting from the first to the second part

Second part (8 transparencies)

Fisher information is nonlinear on resources $F \propto \bar{n}^2$

This allows to reach strong Heisenberg limit for large number N of runs in **linear schemes** (suitable for weak probes)

$$G = a^+ a \qquad \Delta\tilde{\theta}_{\min} \approx \frac{1}{2 \quad \bar{n}}, \text{ for } \quad >> 1$$

Exploiting nonlinearity of estimators, even for linear schemes

$$\Delta\tilde{\theta} = \frac{1}{\sqrt{F}} \qquad G = a^{+}a \qquad F \propto \bar{n}^{2} \qquad \bar{n} = \langle\psi|a^{+}a|\psi\rangle$$

Probe close to the vacuum $|\psi\rangle = \mu_{\nu}|0\rangle + \nu|\psi_{\nu}\rangle \qquad \nu << 1, \quad \mu_{\nu} \approx 1$

$|0\rangle = \text{vacuum}$

$|\psi_{\nu}\rangle$ strongly squeezed state / far from the vacuum

$$\langle\psi_{\nu}|a^{+}a|\psi_{\nu}\rangle = \frac{\bar{n}}{\nu^{2}} >> 1$$

$$\bar{n} = \langle\psi|a^{+}a|\psi\rangle = \text{independent of } \nu$$

mean photon number can be small,
(in accordance with the small photon number of nonclassical states)

Coherent version of the proposal in

PRL **104,** 103602 (2010) PHYSICAL REVIEW LETTERS week ending 12 MARCH 2010

Quantum Metrology with Two-Mode Squeezed Vacuum: Parity Detection Beats the Heisenberg Limit

Petr M. Anisimov,* Gretchen M. Raterman, Aravind Chiruvelli, William N. Plick, Sean D. Huver, Hwang Lee, and Jonathan P. Dowling

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(Received 4 November 2009; published 12 March 2010)

Coherence may more effectively improve resolution

homodyne detection, signal= phase shift

$$P(x\,|\,\theta) = \left|\left\langle x \middle| e^{i\theta a^{+}a} \middle| \psi \right\rangle\right|^{2}$$

|x> eigenstates of quadrature X

retaining just terms linear in v

$$p(x\,|\,\theta) \approx \frac{1}{2\pi}\left(e^{-x^{2}/2} + \frac{2\nu}{\sqrt{\Delta X_{\nu}}}\cos\left(\bar{Y}_{\nu}x/2\right)e^{-x^{2}/4}\exp\left[-\frac{(x-\bar{Y}_{\nu}\theta)^{2}}{4(\Delta X_{\nu})^{2}}\right]\right)$$

$$(\Delta X_{\nu})^{2} = \frac{\nu^{2}}{2\bar{n}}$$

X variance in state $|\psi_{\nu}\rangle$

$$\bar{Y}_{\nu} = \frac{\sqrt{2\bar{n}}}{\nu}$$

mean Y in state $|\psi_{\nu}\rangle$

$$F \approx \frac{\nu^{2}\bar{Y}^{2}}{(\Delta X_{\nu})^{2}} = \frac{4\bar{n}^{2}}{\nu^{2}}$$

$$\Delta\tilde{\theta} = \frac{\nu}{2\sqrt{\bar{n}}} \overset{?}{<} \text{weak Heisenberg limit} = \frac{1}{2\sqrt{\bar{n}}}$$

$$\Delta\tilde{\theta} = \frac{\nu}{2\sqrt{\bar{n}}} \overset{?}{<} \text{strong Heisenberg limit} = \frac{1}{2\bar{n}}$$

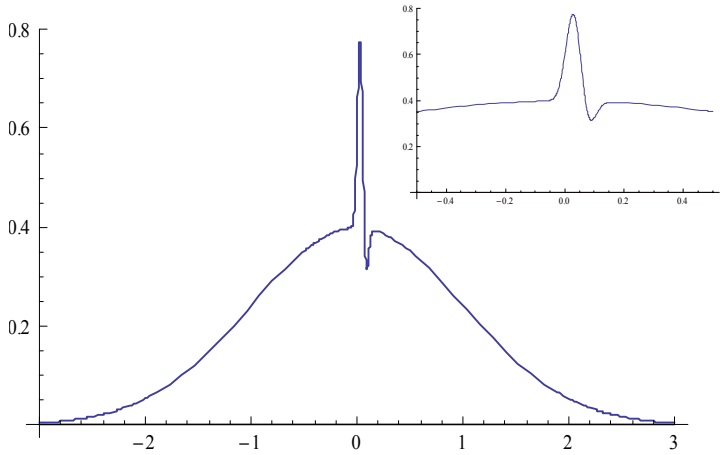
Two issues

§ Efficiency

§ Prior

First issue: efficiency = how many repetitions?

small peak → many repetitions N are necessary to locate the peak



$$\nu = 0.1, \quad \bar{n} = 10, \quad \theta = 0.001$$

Number of measurements N required to locate the peak with probability of error η
Kullback-Leibler divergence S

$$e^{-NS} \leq \eta$$

For distributions close to the vacuum $p_{vac}(x)$

$$S = \int dx \left[p_{\text{vac}}(x) + \delta p(x) \right] \ln\left(1 + \frac{\delta p(x)}{p_{\text{vac}}(x)} \right) \approx \int dx \, \delta p(x) = \text{peak area} \rightarrow \quad > \frac{\ln(1/\eta)}{\sqrt{2}} \left(\frac{\pi}{2} \right)^{1/4} \frac{\bar{n}^{1/4}}{\nu^{3/2}}$$

Second issue: final uncertainty smaller than prior uncertainty?

Ziv-Zakai bound

M. Tsang arXiv: 111.3568 [quant-ph] rectangular prior of width W

$$\left(\Delta\tilde{\theta}\right)^2_{ZZ} = \frac{1}{2}\int_0^W d\tau\tau\left(1-\frac{\tau}{W}\right)\left(1-\sqrt{1-\left|\left\langle\psi\left|\exp\left(i\tau a^+ a\right)\right|\psi\right\rangle\right|^2}\right)$$

$\Delta\tilde{\theta}_{ZZ}$ better than prior uncertainty W provided that $\nu^2 \geq 1$

Beating the strong form of the Heisenberg limit?

$\nu^2 \geq 1$ doing much better than prior precludes beating strong Heisenberg limit

$$\Delta\tilde{\theta} = \frac{\nu}{2\sqrt{\bar{n}}} > \frac{1}{2\bar{n}} = \text{strong Heisenberg limit}$$

Recent confirmations (september 2011-january 2012) of the strong form of Heisenberg limit for linear schemes under different prior asumptions. Universal confirmation?

V. Giovannetti, S. Lloyd, L. Maccone, arXiv: 1109.5661

M. J. W. Hall, D. W. Berry, M. Zwierz, H. M. Wiseman, arXiv: 1111.0788

M. Tsang arXiv: 1111.3568v3

V. Giovannetti, L. Maccone, arXiv:1201.1878v1

Beating the weak (practical) form of the Heisenberg limit? Approaching strong form with N>>1

Example

$$\eta = 0.05, \quad \bar{n} = 10, \quad \nu = 0.01, \qquad = 10^5, \quad W = 10^{-3}$$

small peak

better than prior

$$= 10^5 > \frac{\ln(1/\eta)}{\sqrt{2}} \left(\frac{\pi}{2}\right)^{1/4} \frac{\bar{n}^{1/4}}{\nu^{3/2}} = 4 \times 10^3 \qquad \nu^2 = 10 > 1$$

$$\Delta\tilde{\theta} = \frac{\nu}{2\sqrt{\bar{n}}} = 1.6 \times 10^{-6} \qquad \Delta\tilde{\theta}_{ZZ} = 1.7 \times 10^{-6}$$

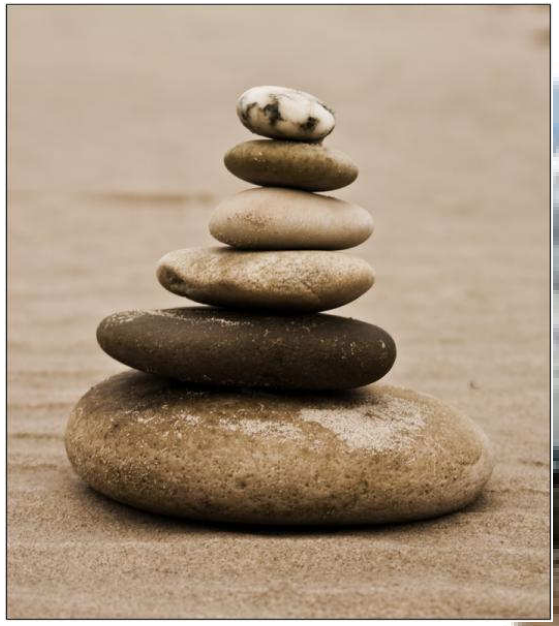
$$\Delta\tilde{\theta}_{\text{strong}} = \frac{1}{2\bar{n}} = 5 \times 10^{-7} \qquad \Delta\tilde{\theta}_{\text{weak}} = \frac{1}{2\sqrt{\bar{n}}} = 1.6 \times 10^{-4}$$

Very close to strong Heisenberg limit with small photon numbers

$$\Delta\tilde{\theta} \approx \Delta\tilde{\theta}_{ZZ} \approx 3\Delta\tilde{\theta}_{\text{strong}} << \Delta\tilde{\theta}_{\text{weak}}$$

For standard squeezed case (υ=1) same N and $\bar{n}$ $\Delta\tilde{\theta}_{ZZ} = 1.6 \times 10^{-4}$

Brief relaxation
while shifting from the second to the third (last) part



Third part **(9 transparencies)**

Typical wisdom: Quantum limits arise from uncertainty relations

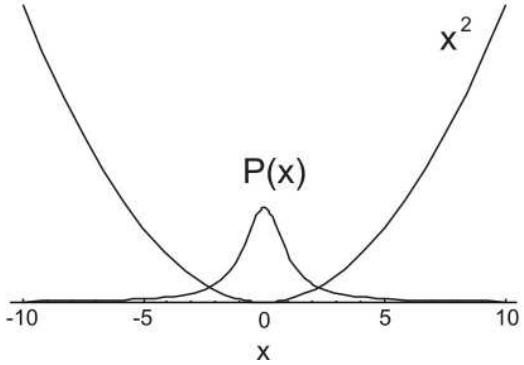
Typically, uncertainty is measured using variance.

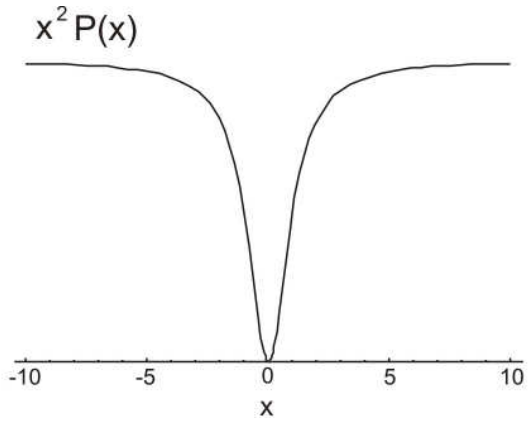
This is not the only possibility.

Alternative assessments of uncertainty:
contradictions and lack of uncertainty?

DIFFICULTIES OF VARIANCE

✓Too much weight on extreme, unlikely values of the variable (leading to divergences in simple situations)





✓ Not practical for phase/angular variables because of periodicity

✓ Trivial uncertainty relation in finite-dimensional systems

$$\Delta A \Delta B \geq 0$$

ΔA can be zero while ΔB is always finite

ALTERNATIVE FAMILIES OF MEASURES

Exponential of Renyi entropies $$R_q(X) = \left\{ \int dx \left[P(x) \right]^q \right\}^{\frac{1}{1-q}}$$

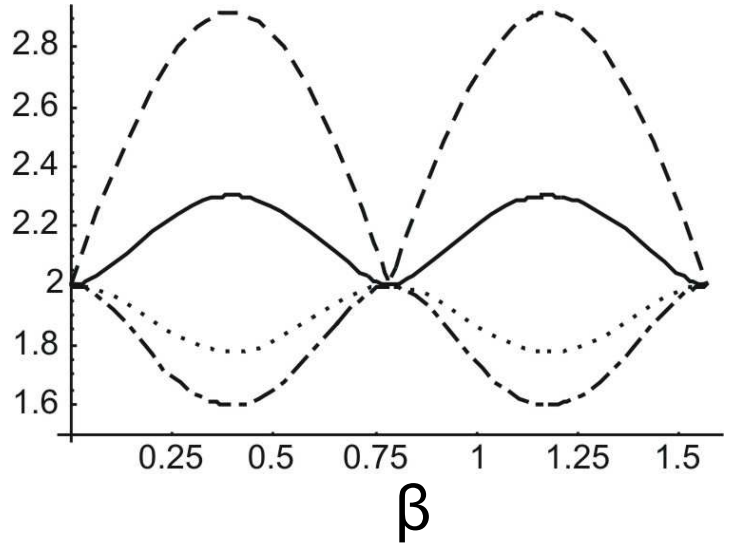
Tsallis entropies $$S_q(A) = \frac{1 - \int dx \left[P(x) \right]^q}{q-1}$$

P(x) = statistics of observable X q = real parameter

spin ½ example

$$|\psi\rangle = \cos\beta|\uparrow\rangle + \sin\beta|\downarrow\rangle$$

$$R_q(\sigma_z)R_q(\sigma_x)$$



from top to bottom
q=0.5
q=1
q=2
q=3

states β=π/8 have **maximum** joint uncertainty for q=0.5, 1

states β =π/8 have **minimum** joint uncertainty for q=2, 3

Fully contradictory results

Physica A 387 (2008) 4800–4808

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

Some extensions of the uncertainty principle

Steeve Zozor [a,*], Mariela Portesi [b], Christophe Vignat [c]

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Joint uncertainty of complementary observables may have no lower bound.

$$R_2(x)R_2(p) \to 0$$

This suggests that measures of uncertainty alternative to variance in quantum metrology may lead to:

i) contradictory conclusions

ii) no quantum limits

Covariant detection of displacements $P(x|\theta) = P(x-\theta)$

Distance between the statistics before $P(x)$ **and after** $P(x-\theta)$ **the displacement** θ

$$D^2 = \frac{1}{2}\int_{-\infty}^{\infty} dx\left([P(x-\theta)]^q - [P(x)]^q\right)^{1/q}$$

For small signals $D^2 \approx \theta^{1/q} F(q)$

$$F(q) = \frac{q^{1/q}}{2}\int_{-\infty}^{\infty} dx[P(x)]^{(q-1)/q}\left[\frac{dP(x)}{dx}\right]^{1/q} = \frac{q^{1/q}}{2}\int_{-\infty}^{\infty} dxP(x)\left[\frac{dP(x)/dx}{P(x)}\right]^{1/q}$$

generalized Fisher information

F determines the visibility of the displacement

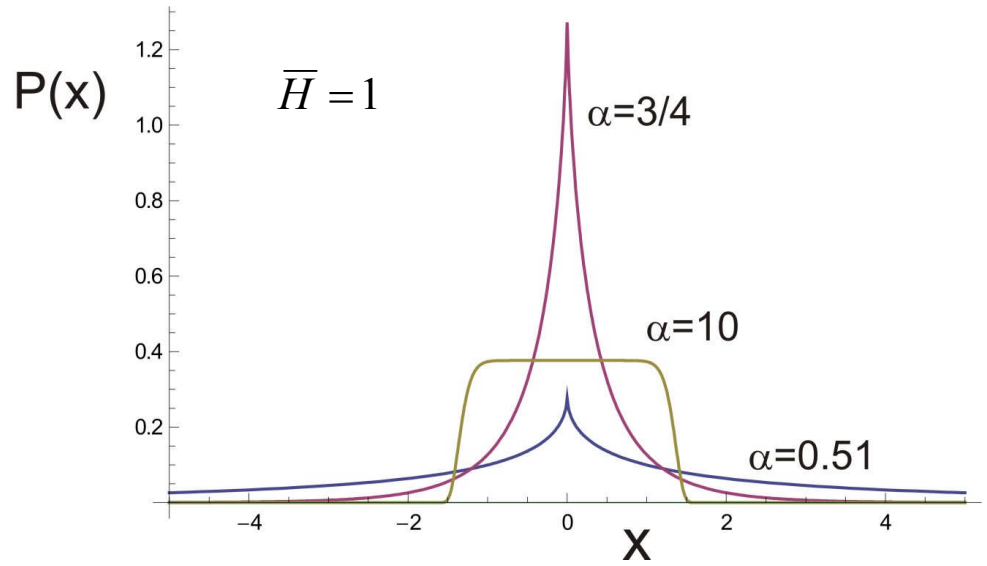
Larger means F larger resolution: minimum signal $\theta_{\min} \propto \frac{1}{F^q(q)}$

For q=1/2 this is Fisher information and Cramér-Rao bound

Probe = free particle in exponential state

$$P(x) = \frac{\alpha 2^{1/\alpha}}{2\gamma\Gamma(1/\alpha)} \exp\left(-2|x/\gamma|^{\alpha}\right)$$

α, γ state parameters

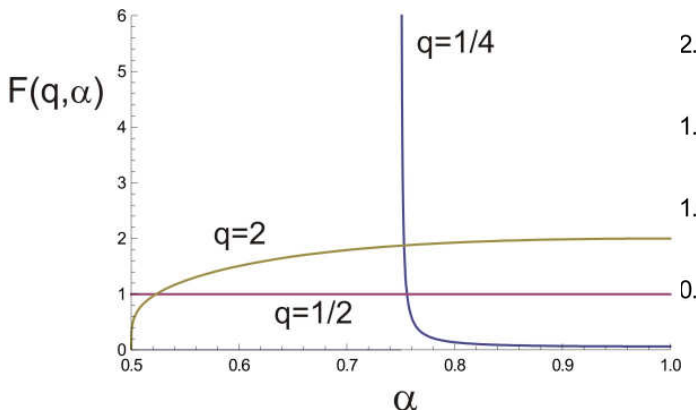


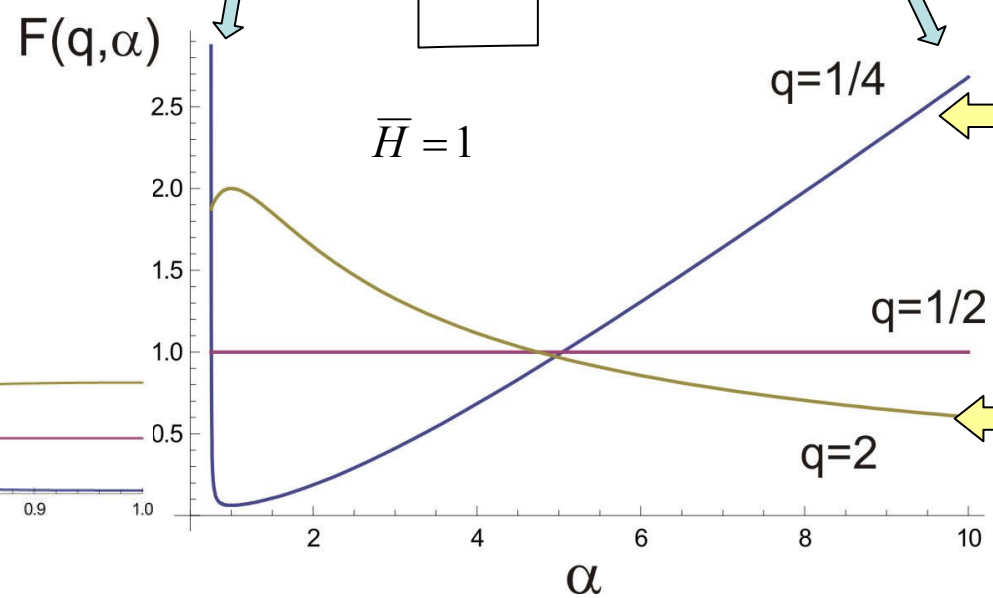
$$\overline{H} = \text{mean energy free particle} = \overline{p^2} = (\Delta p)^2 = \frac{\alpha^2 2^{2/\alpha}}{4\gamma^2} \frac{\Gamma(2-1/\alpha)}{\Gamma(1/\alpha)}$$

$$F(q,\alpha) = \frac{\Gamma\left(\frac{\alpha+q-1}{\alpha q}\right)}{\Gamma(1/\alpha)} \left[4q^2 \overline{H} \frac{\Gamma(1/\alpha)}{\Gamma(2-1/\alpha)}\right]^{1/(2q)} \qquad \alpha + q > 1 \qquad \alpha > 1/2$$

$$F(q,\alpha)=\frac{\Gamma\left(\dfrac{\alpha+q-1}{\alpha q}\right)}{\Gamma(1/\alpha)}\left[4q^2\overline{H}\,\frac{\Gamma(1/\alpha)}{\Gamma(2-1/\alpha)}\right]^{1/(2q)}$$

Arbitrary large resolution as $\alpha\to 1\text{-}q$ or ∞ (for any fixed mean energy)





Contradiction between different values for q

Summarizing

§ Nonlinearity improves quantum resolution via two ways:

(i) Generate nonclassical probes

(ii) Imprint signal information

The optimum way of using nonlinearity is (ii) . Three main features:

(a) uncertainty decreases below the Heisenberg limit

(b) classical robust probes can be used

(c) resolution depends on new variables

§ Nonlinearity of performance estimators may be used to beat weak form of Heisenberg limit with very weak probes approaching the strong limit

§ Different performance measures may lead to contradictory conclusions and arbitrarily high resolution not limited by quantum mechanics

These results are contained in :

Review on nonlinear quantum metrology
Quantum-limited metrology with nonlinear detection schemes
A. Luis**,** SPIE reviews **1,** 018006 (2010)

Challenging metrological limits via coherence with the vacuum
Á. Rivas and A. Luis, arXiv:1105.6310

Precision quantum metrology and nonclassicality in linear and nonlinear detection schemes
Á. Rivas and A. Luis, Phys. Rev. Lett. **105**, 010403 (2010)

Effect of fluctuation measures on the uncertainty relations between two observables: Different measures lead to opposite conclusions
A. Luis, Phys. Rev. A **84**, 034101 (2011)

Alternative measures of metrological resolution: contradictions and lack of limits
A. Luis and A. Rodil, arXiv:1201.3072