# Quantum limits in phase estimation

## The role of adaptive measurements and Fisher information

H. M. Wiseman, B. L. Higgins, G. J. Pryde, D. W. Berry,
S. D. Bartlett, and Others

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CENTRE FOR QUANTUM COMPUTATION & COMMUNICATION TECHNOLOGY
AUSTRALIAN RESEARCH COUNCIL CENTRE OF EXCELLENCE

# Outline


1. Quantum Phase Estimation
2. Single-Pass Experiments Beyond the SQL
3. Multi-Pass Experiments Scaling at the HL
4. Qubit Gate Characterization Experiment and Theory
5. Conclusions

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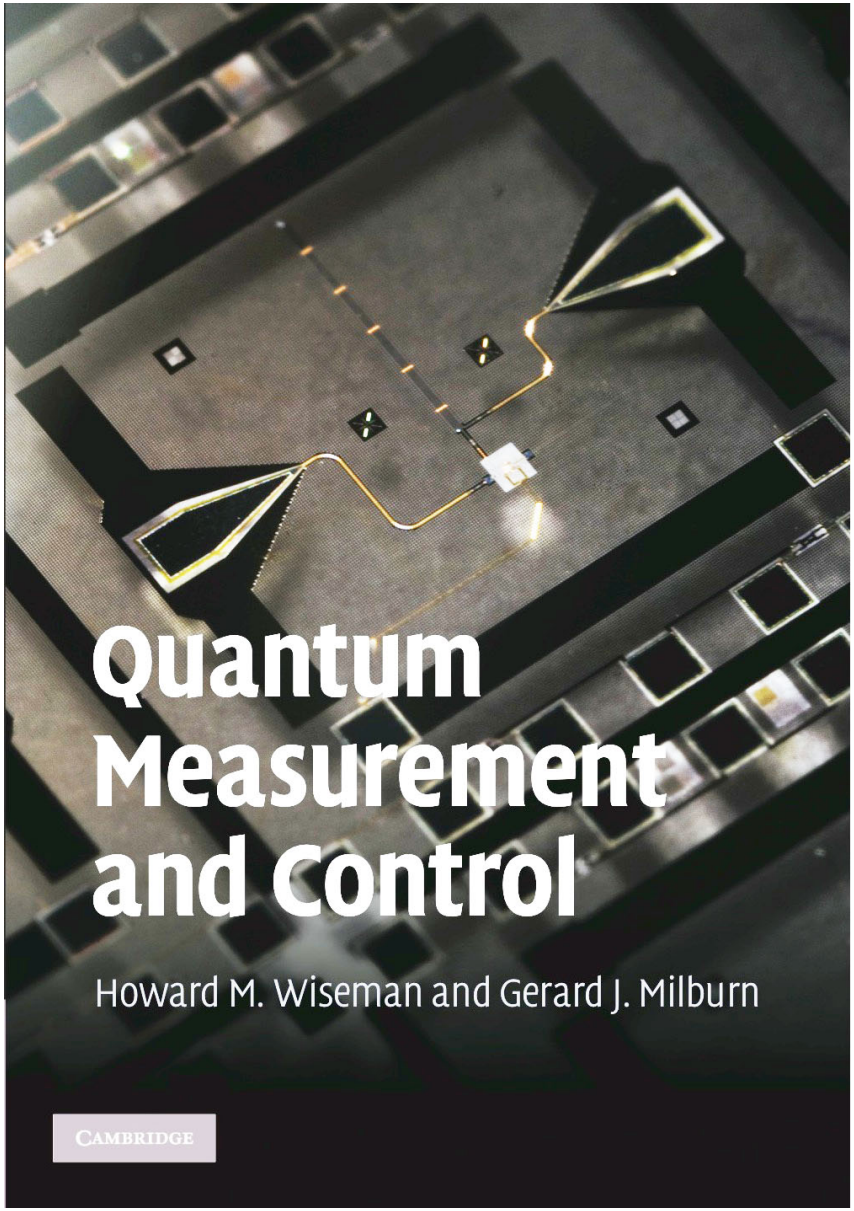
# Quantum Measurement and Control




- Quantum measurement theory puts fundamental limits on the accuracy of phase estimation $\Longrightarrow$ quantum phase estimation.
- measurement-based quantum control is making the unitary evolution of, *or* measurements on, the system depend upon the results of past measurements.
- The latter case, adaptive measurements, can help to achieve the quantum limits in quantum phase estimation.

# What is a quantum phase?

- A phase $\phi$ is an argument in a unitary operator

$$\hat{U} = \exp(i\phi\hat{n}),$$

  where the spectrum of $\hat{n}$ is (a subset of) the naturals $\{0, 1, 2, \cdots\}$.
- e.g. $\phi$ could be a linear optical phase shift and $\hat{n}$ could be
  - the photon number operator $\hat{a}^\dagger\hat{a}$ of a field mode.
  - the total number of photons $\sum_{k=1}^{K}\hat{a}_k^\dagger\hat{a}_k$ across a bunch of modes.
  - the total number of *photon passes*: $\sum_{k=1}^{K} p_k\hat{a}_k^\dagger\hat{a}_k$, where $p_k$ is the number of times mode $k$ passes through the phase shift.
- Or $\phi$ could be a *nonlinear* optical phase shift — a $\chi^{(2q-1)}$ self-phase-modulation parameter — and $\hat{n}$ could be
  - the $q$th power of the photon number operator $(\hat{a}^\dagger\hat{a})^q$ of a field mode.
  - blah blah $\sum_{k=1}^{K} p_k(\hat{a}_k^\dagger\hat{a}_k)^q$
- or $\phi$ could be a qubit gate parameter, a case considered later.

# Defining the accuracy of a phase estimate

- Denote the phase to be estimated by $\Phi$, the estimate by $\breve{\Phi}$, and the error $\breve{\Phi} - \Phi \in [-\pi, \pi]$. (Capitals denote random variables.)
- The Mean-Square-Error is $\mathrm{MSE}_\phi := \int_{\phi-\pi}^{\phi+\pi} d\breve{\phi}\, (\breve{\phi} - \phi)^2\, p(\breve{\phi}|\phi)$
- For **phase-sensing** around $\phi$, $\mathrm{MSE}_\phi$ is sufficient. But for *ab initio* phase estimation we need the Average-Mean-Square-Error

$$(\delta\breve{\Phi})^2 = \frac{1}{2\pi}\int_0^{2\pi} d\phi\, \mathrm{MSE}_\phi = \int_{-\pi}^{\pi} d\varphi\, \varphi^2\, \overline{p}(\varphi)$$

where $\overline{p}(\varphi) = \frac{1}{2\pi}\int_0^{2\pi} d\phi\, p(\varphi + \phi|\phi)$, the distribution for $\breve{\Phi} - \Phi$.

- If $\overline{p}(\varphi)$ is symmetric and concentrated about zero, the AMSE is well approximated by the Holevo Variance,

$$(\delta_H\breve{\Phi})^2 := |\langle e^{i\Theta}\rangle|^{-2} - 1,$$

which is often more useful in practice.

# Standard Quantum Limit and Heisenberg Limit

- To obtain limits for phase estimation we must restrict the state, e.g. to one with support on the $\hat{n}$-eigenvalues $\{0, 1, \cdots, N\}$.
- The Standard Quantum Limit is

$$(\delta\check{\Phi})^2_{\text{SQL}} \sim 1/N \text{ for } N \gg 1.$$

- This can be achieved for $\hat{n} = \quad {}_{k=1}^{K}(\hat{a}_k^\dagger \hat{a}_k)$, with $K = N$ and an input state which is a tensor product of states of the form $|0\rangle + |1\rangle$ (where these denote eigenstates of $\hat{a}_k^\dagger \hat{a}_k$), or e.g. $|0, 1\rangle + |1, 0\rangle$.
- The ultimate quantum limit is **what I will call the** Heisenberg Limit [Luis and Peřina, PRA (1996); Hall *et al*, arXiv 1111.0788v1],

$$(\delta\check{\Phi})^2_{\text{HL}} \sim \pi^2/N^2 \text{ for } N \gg 1.$$

- This is the *achievable lower bound* for quantum phase estimation.
- For a general and rigorous bound with $\langle\hat{n}\rangle$ instead of $N$ see Hall, Berry, Zwierz and Wiseman, arXiv 1111.0788v1 (*cf.* the more complicated bound of GLM, which is not limited to phase).

# The (Quantum) Cramér-Rao Bound

- Consider the following special case:
  - The input state is a tensor product of $M$ copies of a given state $\rho$.
  - Each copy $m$ is transformed as $\rho_m \to \exp(i\phi\hat{n}_m)\rho_m \exp(-i\phi\hat{n}_m)$.
  - Each copy is fed into an *identical* measurement apparatus $\mathcal{M}$.
- In this case there exists a Cramér-Rao Bound for the error

$$(\delta\breve{\Phi})^2 \geq \frac{1}{M \times F_{\mathcal{M}}(\phi, \rho)}$$

  where $F_{\mathcal{M}}(\phi)$, the *Fisher information*, is an easily calculable function of $\rho$, $\mathcal{M}$, and the true phase $\phi$.
- Moreover, there is also a *Quantum* Cramér-Rao Bound:

$$(\delta\breve{\Phi})^2 \geq \frac{1}{M \times F_Q(\phi, \rho)},$$

  where $F_Q(\phi) \equiv \max_{\mathcal{M}} F_{\mathcal{M}}(\phi, \rho)$ is an easily calculable (for $\rho$ pure anyway) function of $\rho$ and $\phi$.
- These bounds are often achievable for $M \gg 1$ but *not always*.

# Fisher Information, SQL, and HL

- Recall $(\delta\breve{\Phi})^2_{\rm SQL} \sim 1/N$ for $N \gg 1$.
- Consider again the following special case where
  - the input comprises many ($M$) identical copies, so that $N = M \times n_{\rm max}$, where each copy has support on $\{0, \cdots, n_{\rm max}\}$.
  - each copy is measured identically, so $(\delta\breve{\Phi})^2 \geq 1/[M \times F_{\mathcal{M}}(\phi, \rho)]$.
- Then to surpass the SQL it is *necessary* to have $F_{\mathcal{M}}(\phi, \rho) > n_{\rm max}$.
- It can be shown that $F_Q(\phi, \rho) \leq n_{\rm max}^2$, and this bound is achievable, by the NØØN state $|n_{\rm max}, 0\rangle + |0, n_{\rm max}\rangle$ Jøn Døwlıng talked about.
- However this does not mean that the bound

$$(\delta\breve{\Phi})^2 \geq \frac{1}{M \times n_{\rm max}^2} = \frac{M}{N^2}$$

is achievable (by any state) and in particular it cannot possibly be even close to achievable for $M < 10$, because

$$(\delta\breve{\Phi})^2_{\rm HL} \sim \frac{\pi^2}{N^2}.$$

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# Previous Experiments (to 2010)

- Sub-SQL photonic interferometry experiments go back to Kuzmich & Mandel (1998) [sub-SQL *phase quadrature* measurements (i.e. squeezing, homodyne) are older.]
- Previous experiments have (at best) prepared states $\rho$ and done measurements $\mathcal{M}$ with a particular phase $\phi$ such that

$$F_{\mathcal{M}}(\phi, \rho) > n_{\max},$$

  a *necessary* condition to do sub-SQL phase estimation.
- However, none of them actually estimated a phase by
  1. choosing a random phase $\phi$,
  2. taking a finite data set using state(s) with maximum photon numbers **summing** to $N$ in **total**,
  3. using the data to obtain an estimate $\breve{\phi}$, without knowledge of $\phi$,
  4. calculating the square error $\varphi^2 = (\breve{\phi} - \phi)^2$.

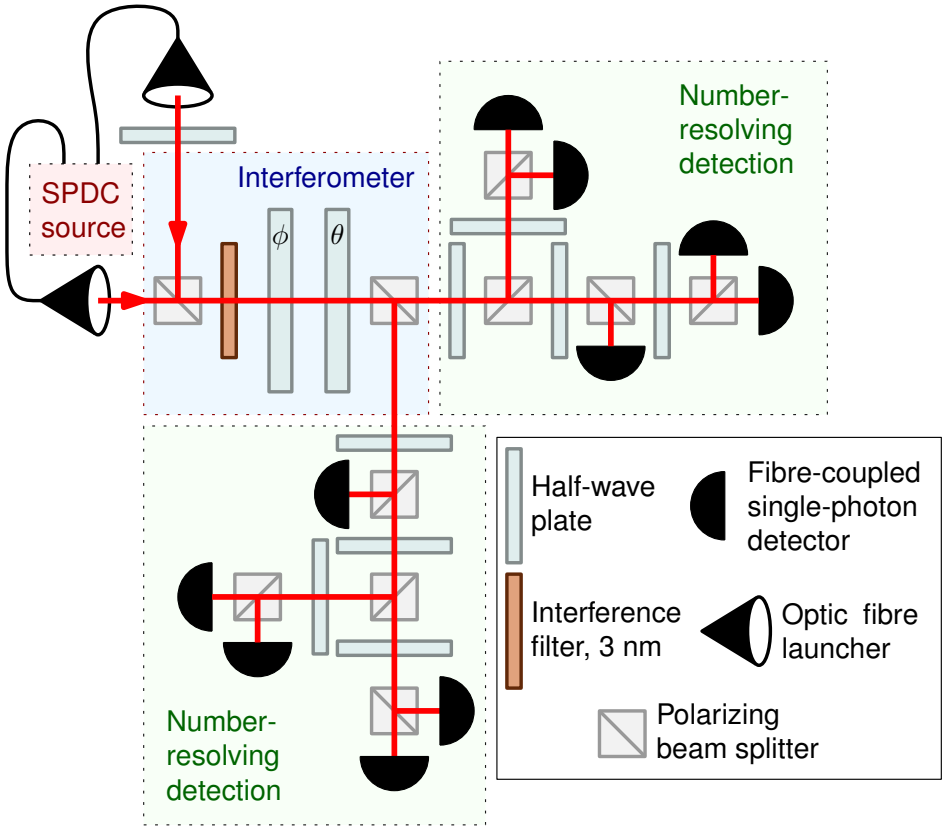
  and repeating many times to obtain an AMSE $(\delta\breve{\Phi})^2$ less than $1/N$.

First *ab initio* sub-SQL (single-pass) phase estimation experiment

nature photonics LETTERS
PUBLISHED ONLINE: 12 DECEMBER 2010 | DOI: 10.1038/NPHOTON.2010.268

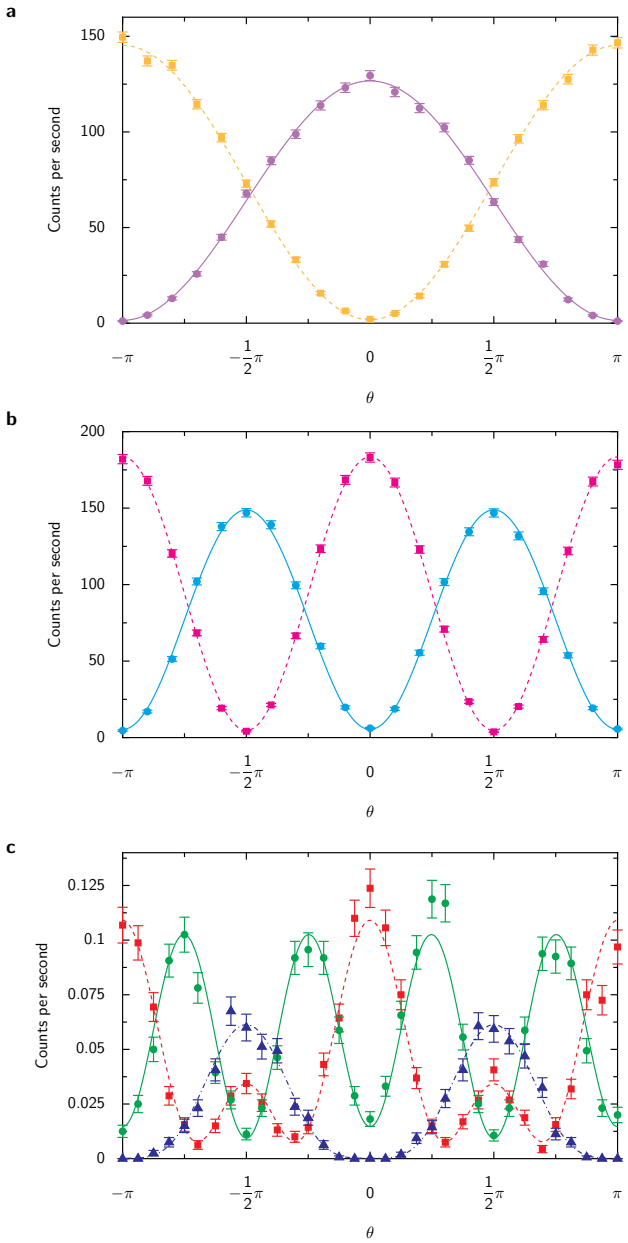
**Entanglement-enhanced measurement of a completely unknown optical phase**

G. Y. Xiang[1,2], B. L. Higgins[1], D. W. Berry[3], H. M. Wiseman[1]* and G. J. Pryde[1]*



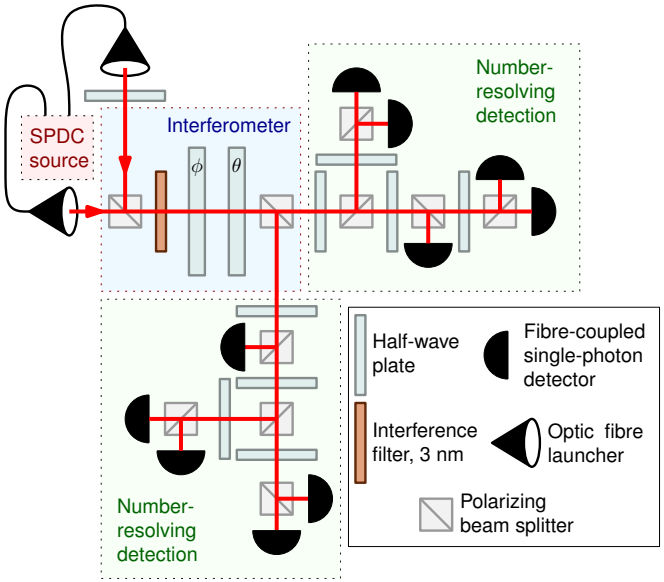

- By *post-selecting* on getting $n = 2$ or 4 photons, our input states are $|\text{Holland-Burnett}\rangle$.
- Unlike other such experiments, this is (essentially) the only post-selection we do.
- i.e. we keep all click-patterns: (0,1) and (1,0) for $n = 1$; (0,2), (1,1), and (2,0) for $n = 2$; (0,4), (1,3), (2,2), (3,1), and (4,0) for $n = 4$.

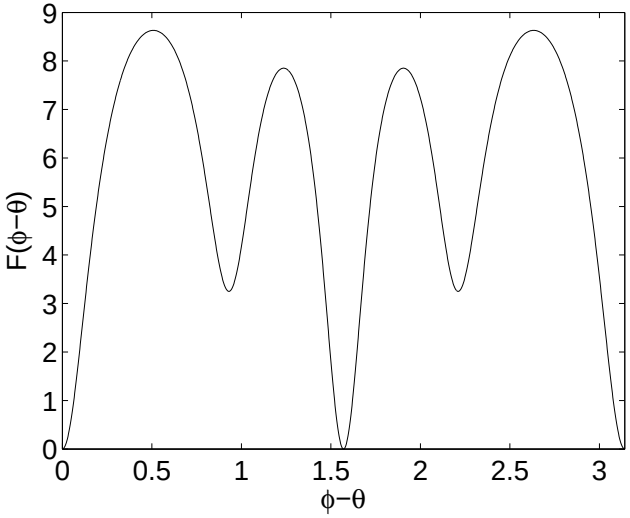
# Why one can't use the same state $\rho$ each time




- There are two 1-photon fringes: (0,1); and (1,0), with period $2\pi$. This gives the SQL.
- There are two 2-photon fringes: (0,2) and (2,0) together; and (1,1), with period $\pi$.
- There are three 4-photon fringes: (0,4) and (4,0) together; (1,3) and (3,1) together; and (2,2). These have periods $\pi$, $\pi/2$, and $\pi$ resp.
- Using only one type of state (2 or 4 photon) makes *ab initio* phase estimation impossible because there is no way to distinguish a phase $\phi$ from a phase $\phi + \pi$ so $(\delta\check{\Phi})^2 \geq (\pi/2)^2$.
- The optimal solution, for a given $N$, is to use some 1-, some 2-, and some 4-photon states.
- e.g. for $N = 48$, we use this $N$-photon sequence: $10\times$2-photon states, then $8\times$1-photon states, then $5\times$4-photon states.

# Why one can't use a fixed measurement $\mathcal{M}$ for each $\rho$



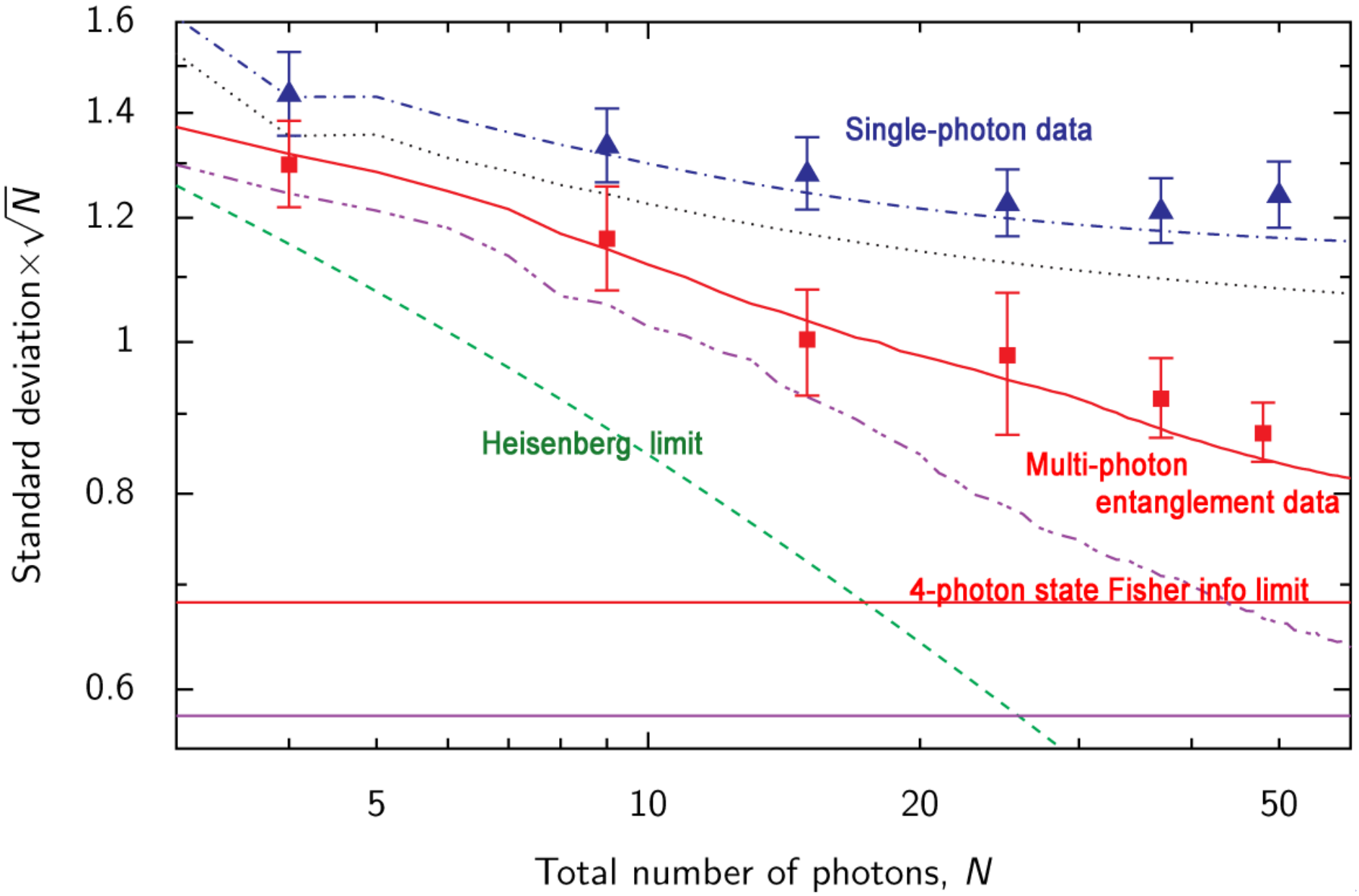




- The measurement $\mathcal{M}$ is defined by photon counting with only a single variable parameter, $\theta$, the auxilliary phase shift.
- The Fisher information

$$F_{\mathcal{M}}(\phi,\rho) \equiv F_\theta(\phi,\rho) = F_0(\phi-\theta,\rho)$$

  is very sensitive to $\phi-\theta$.
- Curve here is a fit from real 4-photon data; recall that sub-SQL requires $F_0(\phi-\theta,\rho) \geq 4$.
- The optimal solution, is to choose $\theta$ depending on $\phi$ so as to maximize $F_0(\phi-\theta)$.
- But $\phi$ is unknown. Therefore we need to use an adaptive technique, in which $\theta$ depends on the estimate of $\phi$ from data from states earlier in the $N$-photon sequence.

# Nevertheless why Fisher information is still useful



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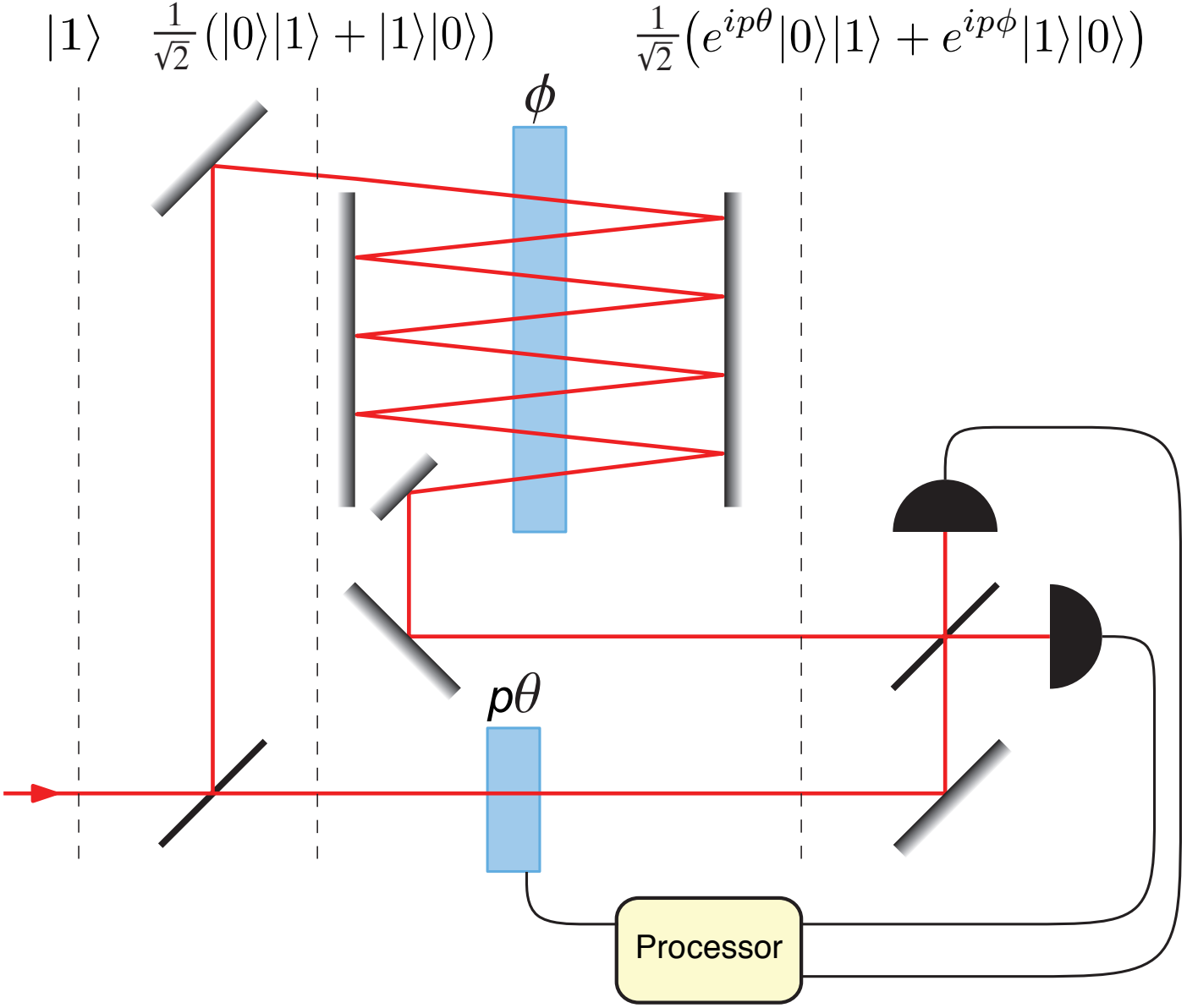
# A single-photon adaptive multipass experiment




- $\hat{U} = \exp(i\phi\hat{n})$, with $\hat{n} = \sum_{k=1}^{K} p_k \hat{a}_k^\dagger \hat{a}_k$.
- There is at most one photon in each mode $k$.
- Thus here $N = \sum_{k=1}^{K} p_k$ is the max. total number of photon-passes through $\phi$.

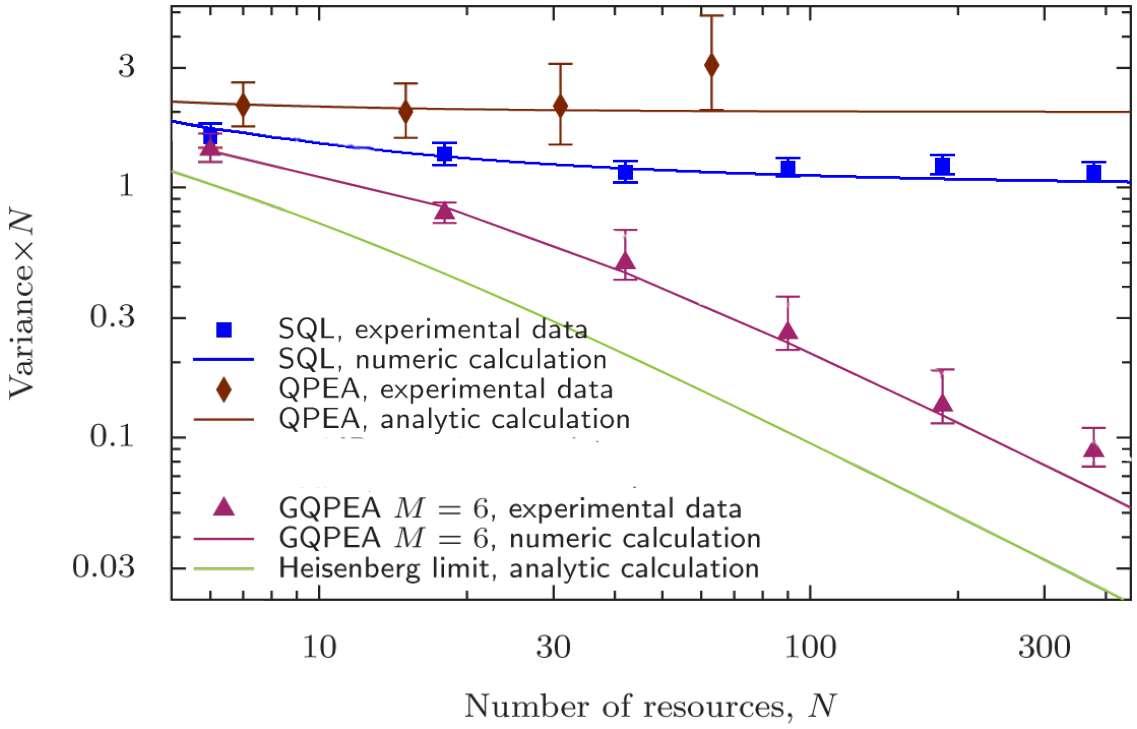
First *ab initio* phase estimation experiment with HL scaling

Vol 450|15 November 2007|doi:10.1038/nature06257 nature

LETTERS

**Entanglement-free Heisenberg-limited phase estimation**

B. L. Higgins[1], D. W. Berry[2], S. D. Bartlett[3], H. M. Wiseman[1,4] & G. J. Pryde[1]




- Multipasses ($p$) give fringes of period $2\pi/p$, just like in $p00p$ states.
- Thus sequences with different $p$ are necessary. e.g. $N = 390$ is 6 each of $32, 16, 8, 4, 2, 1$ passes.
- Adaptive detection is again useful, and this time gives HL-scaling.

Is adaptive detection necessary to achieve HL scaling from multi-pass single-photons?

- As in the adaptive case the sequences have $p = p_k = 2^k \in \{1, 2, \cdots, 2^K\}$.
- In the nonadaptive case we find a good results by repeating each $p_k$

$$M(K, k) = M_K + \mu(K - k)$$

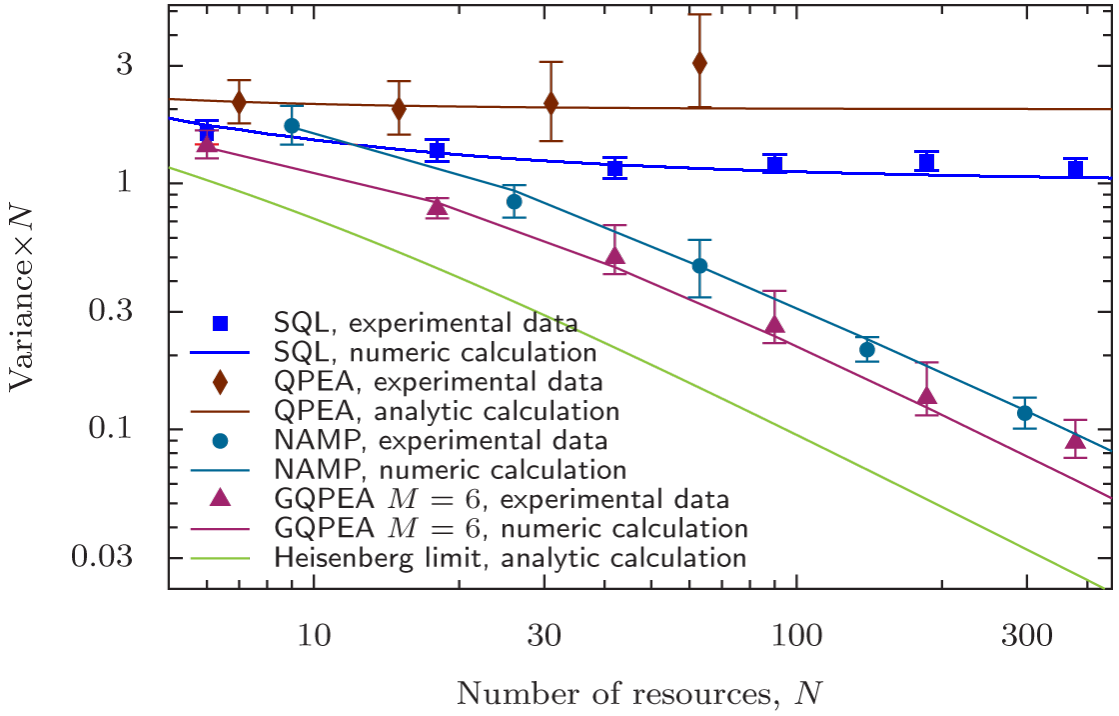
times, with $M_K = 2$, $\mu = 3$.

Is adaptive detection necessary to achieve HL scaling from multi-pass single-photons?

New Journal of Physics
The open-access journal for physics

**Demonstrating Heisenberg-limited unambiguous phase estimation without adaptive measurements**

B L Higgins[1], D W Berry[2,3,4], S D Bartlett[5], M W Mitchell[6], H M Wiseman[1,7] and G J Pryde[1,7]

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# Multipassed single photon $\equiv$ repeated gates on qubit

- For single photon inputs, each pass through an unknown phase is equivalent to the application of a polarization qubit gate

$$\hat{U} = \exp(i\phi\hat{\sigma}_z),$$

where $\hat{\sigma}_z = |1,0\rangle\langle 1,0| - |0,1\rangle\langle 0,1| = |H\rangle\langle H| - |V\rangle\langle V|$.

- If we could make a QND measurement on a photon we could estimate $\phi$ arbitrarily accurately using a single photon.
- For many solid-state qubits QND projective read-out is feasible. However, adaptive measurements are slow.
- Recall our non-adaptive algorithm, with $p_k = 2^k \in \{1, 2, \cdots, 2^K\}$ each $M(K,k) = M_K + \mu(K-k)$ times.

PHYSICAL REVIEW B **83**, 125410 (2011)

**Nanoscale magnetometry using a single-spin system in diamond**

R. S. Said,[1] D. W. Berry,[2] and J. Twamley[1]

This works well even for low-visibility fringes, for suitably chosen $M_K$ and $\mu$.

# Applying our nonadaptive algorithm in magnetometry

nature nanotechnology

LETTERS

PUBLISHED ONLINE: 18 DECEMBER 2011 | DOI: 10.1038/NNANO.2011.224

## High-dynamic-range magnetometry with a single nuclear spin in diamond

G. Waldherr[1]*, J. Beck[1], P. Neumann[1], R. S. Said[2,3], M. Nitsche[1], M. L. Markham[4], D. J. Twitchen[4], J. Twamley[2], F. Jelezko[1,5] and J. Wrachtrup[1]

nature nanotechnology

LETTERS

PUBLISHED ONLINE: 18 DECEMBER 2011 | DOI: 10.1038/NNANO.2011.225

## High-dynamic-range magnetometry with a single electronic spin in diamond

N. M. Nusran, M. Ummal Momeen and M. V. Gurudev Dutt*

# A different resource: Number of *read-outs*.

- If the duration $\tau_{\rm g}$ of the gate $\hat{U}$ is greater than or comparable to the read-out time $\tau_{\rm r}$, it makes sense to quantify resources by

$$N = N_{\rm g} = \text{ number of qubit-passes } \propto \text{ total time .}$$

- However for many qubits $\tau_{\rm r} \gg \tau_{\rm g}$ in which case

$$N_{\rm r} = \text{ number of read-outs } \propto \text{ total time}$$

(Apology: in the plots that follow, $N_{\rm r}$ is simply notated $N$.)

- Further apology: in the following I will use $m$ (rather than $p$) for the number of coherent gate applications prior to a read-out.
- For schemes in which $\{m\}$ is predetermined, $n_m$ is the number of trials using a given $m$, so that $N_{\rm r} = \quad {}^{M}_{m=1}\, n_m$.

# How well can we actually do?

- If we apply the mutlipass-photon protocol presented above we have $m = 1, 2, 4, \cdots 2^K$ and $n_m = 6\,\forall m$, so that $N_{\rm r} = 6(K+1)$ while $N_{\rm g} = 6 \times (2^{K+1} - 1) \approx 6 \times 2^{N_{\rm r}/6}$, so that

$$(\delta\breve{\Phi})^2 \propto N_{\rm g}^{-2} \propto 2^{-N_{\rm r}/3}.$$

- AFAIK the best known scheme is the $n_m = 4\,\forall m$ case of our 2007 Nature scheme, giving

$$(\delta\breve{\Phi})^2 \propto N_{\rm g}^{-2} \propto 2^{-N_{\rm r}/2}.$$

- Is this the best possible scaling? What does the CRB say?
- $F_m(\phi, \psi) = m$. So $F$ can be arbitrarily large, while $N_{\rm r}$ is finite.
- *Fisher information gives no lower bound on* $(\delta\breve{\Phi})^2$ *in terms of* $N_{\rm r}$
- However an informational argument does: each measurement yields at most one bit. At best this can halve the range of $\Phi$, so

$$(\delta\breve{\Phi})^2_{\rm IL} \geq \frac{\pi^2}{3} \times 2^{-2N_{\rm r}}.$$

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# What if we can't change the measurement basis?

- All the above algorithms rely upon an *auxilliary phase* $\theta$ which can be arbitrarily set.
- In the qubit gate characterization case, it may not be possible to do this (e.g. it might be $g$, the coupling of the spin to a magnetic field, which is unknown).
- Therefore it is interesting to ask how well can we do without $\theta$.
- In this case the only thing that can be adapted is $m$, the number of of gate applications prior to read-out.
- Adopting a *locally optimal* adaptive strategy for choosing $m$ gives

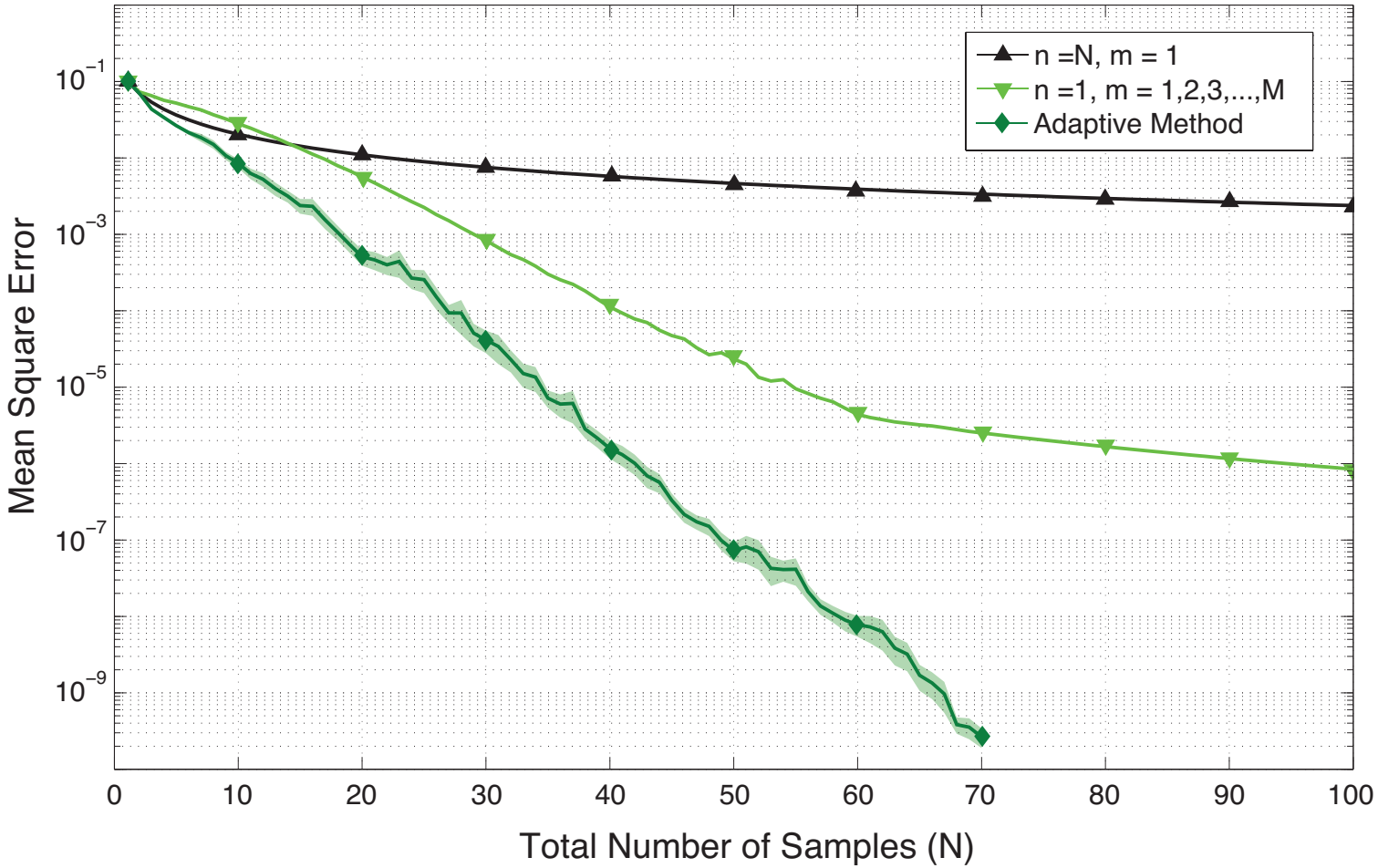
$$(\delta\breve{\Phi})^2 \propto 2^{-\alpha N}, \quad \alpha = 0.21 \pm 0.02$$

and this is exponentially better than standard estimation techniques. [A. Sergeevich, A. Chandran, J. Combes, S. D. Bartlett, and H. M. Wiseman, Phys. Rev. A **84**, 052315 (2011).]

PHYSICAL REVIEW A **84**, 052315 (2011)

**Characterization of a qubit Hamiltonian using adaptive measurements in a fixed basis**

Alexandr Sergeevich,[1] Anushya Chandran,[2,3] Joshua Combes,[2,4] Stephen D. Bartlett,[1] and Howard M. Wiseman[2,*]



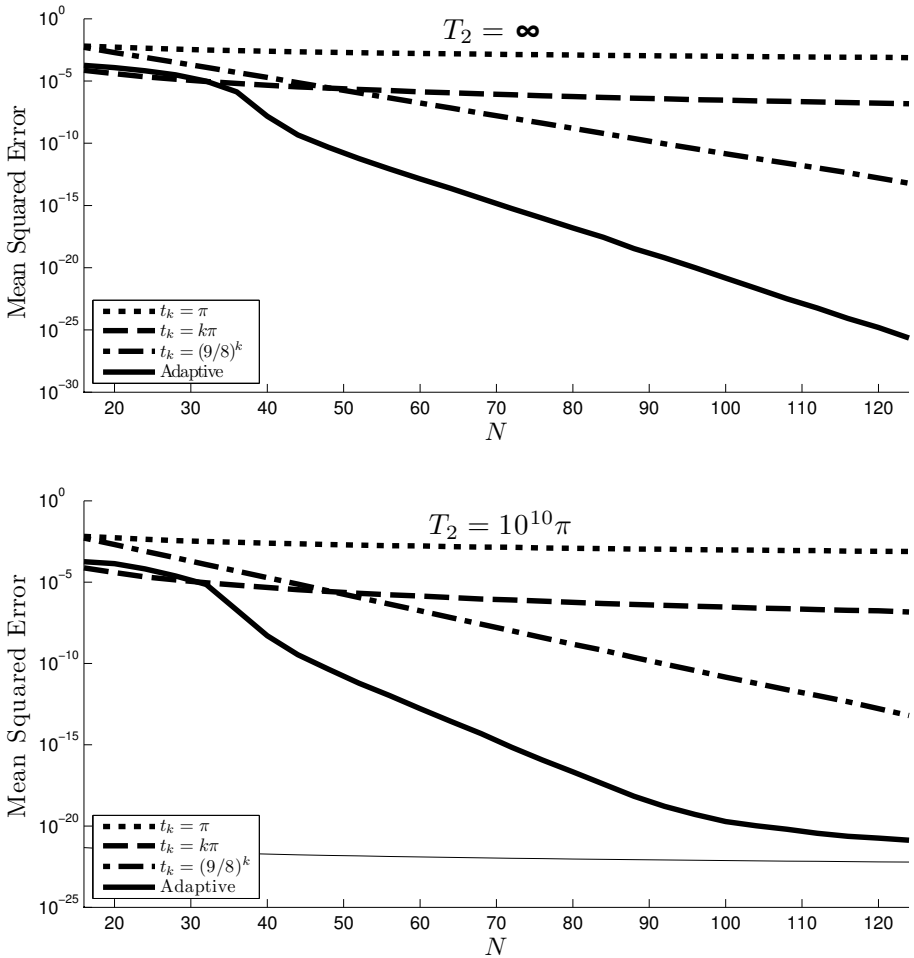

For these non-adaptive schemes, $n \equiv n_m$, so $N_r = n \times \quad {}_m$.

$n = 1$ is near-optimal scheme of this sort, with $(\delta\check{\Phi})^2 \propto N_r^{-3}$.

**How to best sample a periodic probability distribution, or on the accuracy of Hamiltonian finding strategies**

Christopher Ferrie,[1,2] Christopher E. Granade,[1,3] and D.G. Cory[1,4,5]

[1] *Institute for Quantum Computing, University of Waterloo, Waterloo, Ontario, Canada*




This [arxiv:1110.3067 (13 Oct 2011)] reproduced many of our results analytically including:

- For $n = N$ scheme (their "$t_k = \pi$") $(\delta\breve{\Phi})^2 \propto N_{\rm r}^{-1}$
- For $n = 1$ scheme (their "$t_k = k\pi$") $(\delta\breve{\Phi})^2 \propto N_{\rm r}^{-3}$
- For $\pm$ our adaptive scheme, $(\delta\breve{\Phi})^2 \propto 2^{-\alpha N_{\rm r}}$, though $\alpha \approx 0.317$.

Also, using *Fisher information*, they showed for finite $T_2$, $(\delta\breve{\Phi})^2 \gtrsim \frac{\tau_{\rm g}^2}{N_{\rm r} T_2^2}$.

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1. Quantum Phase Estimation
2. Single-Pass Experiments Beyond the SQL
3. Multi-Pass Experiments Scaling at the HL
4. Qubit Gate Characterization Experiment and Theory
5. Conclusions

# Conclusions

- Phase is any parameter that gives a $2\pi$-periodic signal, which turns up in many different contexts.
- Quantum mechanics bounds from below the error of phase estimation (in terms of different resources in different contexts).
- The Cramér-Rao bound (from *Fisher information*) is common, but
  - The bound assumes identical measurements on identical states, which is not the case in many phase estimation schemes.
  - In some cases (e.g. NØØN states) the (Q)CRB is never achievable
  - For some resources (e.g. $N_{\mathrm{r}}$), the CRB gives a trivial lower bound.
- Other bounds (Heisenberg, informational) apply to general scenarios, including sequences of different states, with adaptively changing measurements as needed for optimal phase estimation.
- Nevertheless for many *realistic* situations (e.g. with limited entanglement, limited number of passes, or limited coherence times), the minimum$_\phi$ CRB is a useful lower bound on the error even in these more general scenarios.